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# Cavitation in lubrication. Part 1. On boundary conditions and cavity–fluid interfaces

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The flow of viscous lubricant in narrow gaps is considered for those geometries in which cavitation arises. A detailed review is presented of those boundary conditions which have been proposed for terminating the lubrication regime (i.e. those valid where the cavity forms). Finally it is shown that a uniform cavity–fluid interface remains stable to small disturbances provided that

$$\frac{d}{dx}\left(P+\frac{T}{r}\right) < 0,$$

in which $T$ and $r$ represent the surface tension of the fluid and the radius of curvature of the interface respectively whilst $dP/dx$ is the gradient of fluid pressure immediately upstream of the interface.

## 1. Introduction

This paper is concerned with the phenomenon of cavitation in the context of hydrodynamic lubrication. In the flow within a journal bearing (figure 1*a*) and between counter-rotating cylinders (figure 1*b*) (or between a cylinder and a plane; figure 1*c*), there can arise an air cavity either by ventilation or through dissolved air being expelled from the viscous lubricant once its pressure falls below the saturation vapour pressure.

Over a period of half a century many authors have addressed themselves to the problem of establishing those boundary conditions which are valid at the leading edge of the cavity and which permit a solution to be determined for the pressure distribution in the lubrication regime. With the exception of Floberg (1957) each author has considered a uniform cavity, by which is meant a cavity having a straight-line cavity–fluid interface when viewed from above in a geometry of infinite width. (In the case of a cylinder–plane geometry of infinite width, figure 1(*c*) represents a characteristic cross-section.) Experimentally it is found that such a situation pertains only under restricted conditions if at all; more likely is for the cavity–fluid interface to exhibit a waviness whose wavenumber varies with certain physical parameters (figure 2).

For this and other reasons the last word on the subject of boundary conditions has not been written and it is therefore appropriate to begin this paper with a review of important work and results. Subsequently a criterion is derived according to which a cavity–fluid interface may be uniform or wavy. The implications of this criterion are then examined for each of the cylinder–plane geometry, counter-rotating rollers and a journal bearing. The conclusions reached for the first two geometries support observations that a uniform cavity may arise under restricted conditions. In a closed

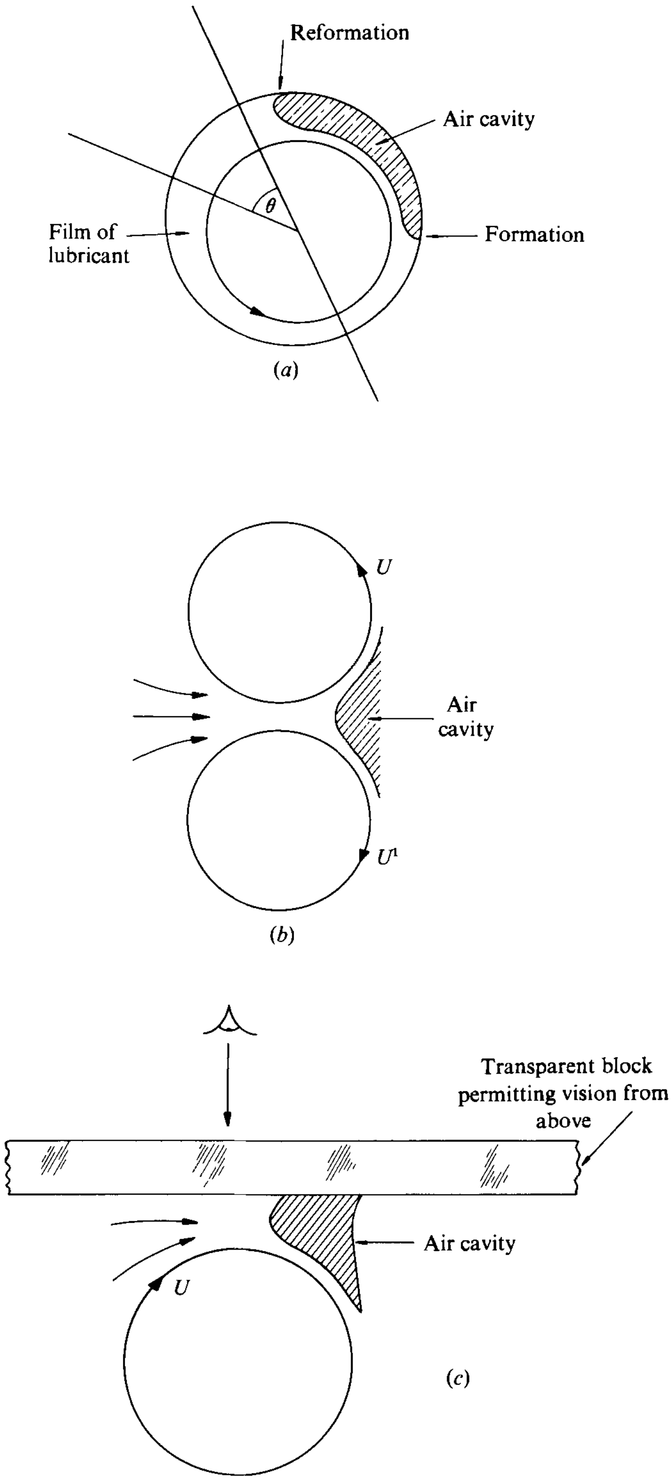


FIGURE 1. Showing the presence of a cavity in cross-sections of (a) a journal bearing, (b) counter-rotating cylinders and (c) a cylinder–plane geometry.

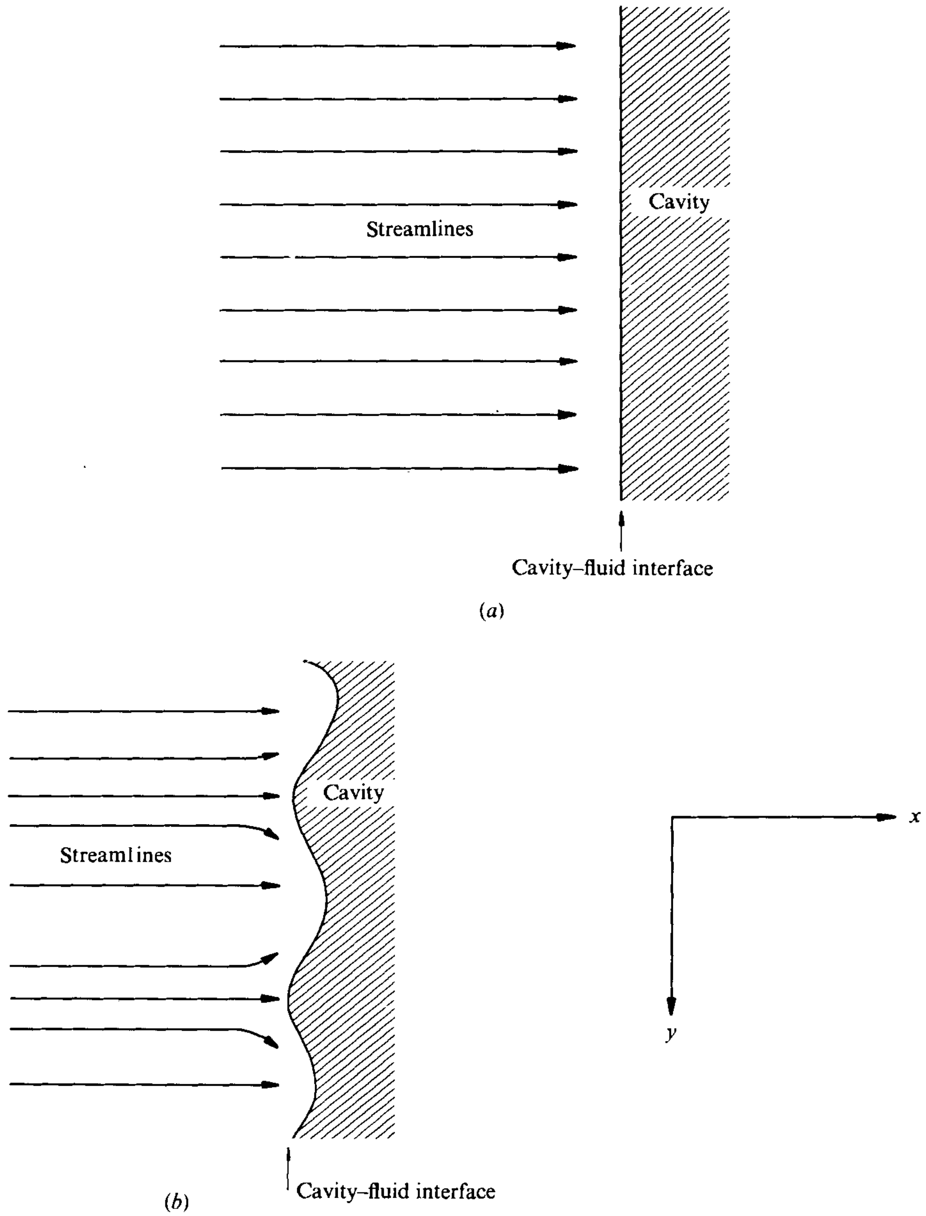


FIGURE 2. $x, y$ cross-sections of flow approaching ($a$) a straight and ($b$) a wavy cavity–fluid interface.

system however, i.e. the journal bearing (figure 1$a$), there are two cavity–fluid interfaces normal to the fluid motion, occurring where the cavity forms and reforms respectively. It is found that under those conditions for which lubrication theory remains valid the criterion for a uniform interface at cavity formation cannot be satisfied whilst at reformation the criterion is essentially reversed and a uniform interface is predicted in agreement with experiment.

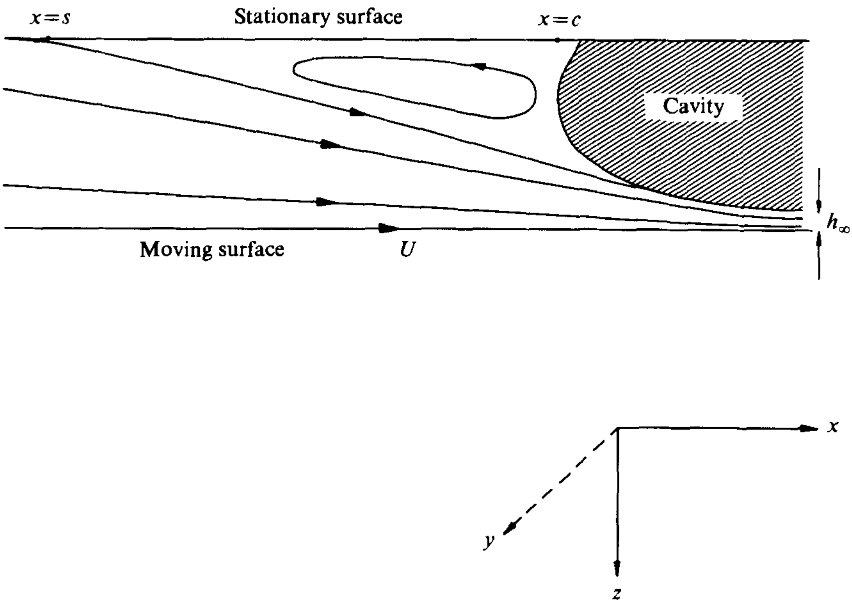


FIGURE 3. An $x, z$ cross-section of the flow near the leading edge of the cavity in a cylinder–plane geometry of infinite width.

## 2. Basic model: review of boundary conditions

Figure 3 represents an enlargement of the flow pattern in the vicinity of the cavity, $x = c$, showing the presence of an eddy downstream of the position of separation $x = s$. The fluid not entrained within the eddy flows beneath the cavity and eventually forms a uniform layer of thickness $h_\infty$ and moves with uniform speed $U$. Rectangular axes are chosen such that $z$ measures distance across the gap $h(x)$ whilst the $x$ and $y$ axes are along the direction of fluid motion and across the width of the bearing respectively.

For steady non-inertial fluid flow, the pressure $P(x, y)$ in the lubrication regime (which extends as far as the cavity) is determined via the familiar Reynolds equation:

$$\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^3 \frac{\partial p}{\partial y}\right) = 6\eta\, U \frac{\partial h}{\partial x}, \tag{2.1}$$

in which $\eta$ and $U$ represent the fluid viscosity and the speed of the moving surface respectively. In a bearing of infinite width which exhibits a uniform cavity gradients with respect to $y$ are identically zero and hence

$$\frac{\partial p}{\partial x} = \frac{6\eta U}{h^2} + \frac{\text{constant}}{h^3}.$$

The velocity distribution in the gap, which satisfies the conditions $u = 0$ on $z = 0$ and $u = U$ on $z = h(x)$, is given by

$$u(x, z) = \frac{1}{2\eta}\frac{\partial p}{\partial x}(z^2 - zh) + \frac{Uz}{h}. \tag{2.2}$$

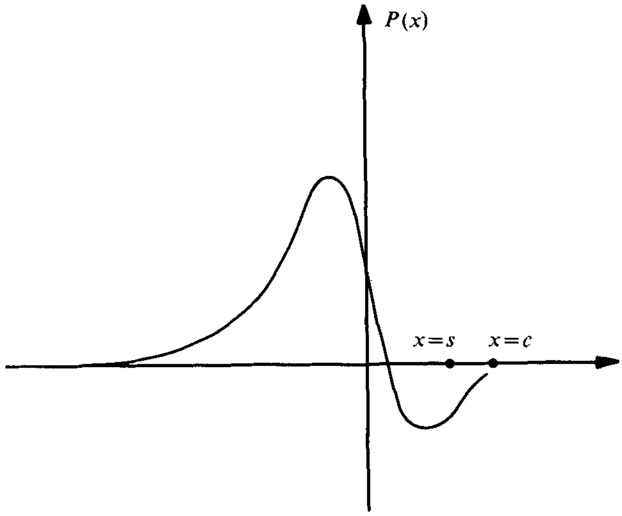


FIGURE 4. The pressure distribution in a lightly loaded cylinder–plane geometry obtained experimentally.

Finally continuity of flow demands that

$$\int_0^{h(x)} u\,dz = Uh_\infty,$$

which in turn reduces to
$$\frac{\partial p}{\partial x} = \frac{6\eta U}{h^2(x)}\left(1 - 2\frac{h_\infty}{h(x)}\right). \tag{2.3}$$

In hydrodynamic lubrication the procedure is to solve the above differential equation subject to appropriate boundary conditions on $P(x)$, namely

$$P(x = -\infty) = 0, \quad P(x = c) = -T/r, \tag{2.4}$$

where $T$ represents the surface tension of the fluid and $r$ the radius of curvature of the cavity–fluid interface in the $x, z$ plane at $x = c$. Unfortunately, the position of the cavity, $x = c$, is unknown; hence in order to obtain a solution a further condition is required, and it is precisely in the formulation of this condition that differences appear in the work of various authors. It is useful to consider a particular geometry, the 'lightly loaded' cylinder–plane geometry, for which the pressure distribution has been found experimentally (figure 4), and see to what extent the various conditions and solutions are able to account for the known features.

*Swift–Stieber condition* (1932, 1933)

Swift, on the basis of a spurious stability argument, and Stieber, using an invalid continuity of flow argument, independently arrived at the condition

$$(\partial p/\partial x)_{x=c} = 0, \tag{2.5}$$

which gives rise to the pressure distribution shown in figure 5. A comparison of this figure with figure 4 reveals that the inadequacy of the Swift–Stieber condition lies in an inability to ascertain the presence of a subcavity pressure loop immediately upstream

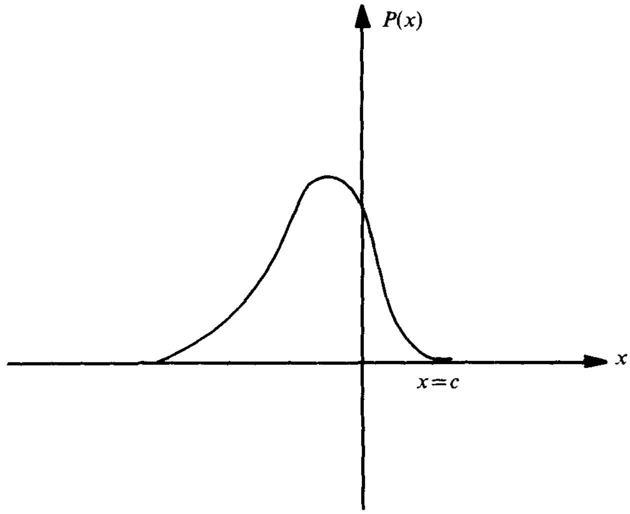


FIGURE 5. A theoretical pressure curve in a lightly loaded cylinder–plane geometry incorporating the Swift–Stieber condition.

of the cavity. Nor indeed does it predict an eddy upstream of the cavity, the existence of which was verified experimentally by Van Der Bergh (1974).

To be fair, however, Floberg (1957) has confirmed that the Swift-Stieber condition is appropriate in 'highly loaded bearings', in which the cavity formation mechanism is somewhat different. In such circumstances high loads would tend to generate high pressures and whilst positive pressures (above atmospheric) are permissible there is a limit to the magnitude of the sub-atmospheric pressure $P_m$ which can be achieved. As the pressure in some part of the fluid decreases to $P_m$, as the load is increased, the fluid film ruptures and air bursts out of solution. This causes a cavity to form further upstream than would otherwise be the case and for this cavity the following conditions apply:

$$P(x = c) = -P_m, \quad (\partial p/\partial x)_{x=c} = 0. \tag{2.6}$$

This kind of cavitation, which is not the subject of this paper, is called vaporization cavitation.

*Hopkins separation condition* (1957)

Several workers (amongst whom Hopkins was the one whose work first appeared in print) have suggested that it is the reverse-flow eddy which provides a mechanism for cavity formation and consequently one should expect the cavity to form where fluid separates from the stationary surface, i.e. where the velocity and tangential stress are both zero (figure 3). A condition on the pressure gradient then follows via the velocity distribution (2.2):

$$(\partial p/\partial x)_{x=c} = 2\eta U/h^2(c), \tag{2.7}$$

which in turn yields a pressure distribution similar to that in figure 4, in which the positions $x = c$ and $x = s$ become coincident. Essentially this implies that the cavity forms at the position of separation and that if there is no separation there is no cavity. For lightly loaded systems this condition provided a major step forward in explaining not only the subcavity pressure loop but also why cavities are always present in cylinder–plane geometries whilst in journal bearings a necessary condition is approximately $e > 0{\cdot}3$, where $e$ is the eccentricity ratio. The explanation follows, of course, from the fact that lightly loaded cylinder–plane geometries always experience separation whilst $e > 0{\cdot}3$ is the criterion for separation in journals.

Unfortunately, by making the position of the cavity coincident with the position of separation, the Hopkins condition does not permit the existence of an eddy (figure 3) and therefore introduces errors which in some circumstances can be appreciable.

### *Taylor* (1963)

Although he did not formulate a condition of his own, Taylor did provide insight and relationships which may well have inspired the later work of Coyne & Elrod (1970, 1971). Following the work of Bretherton (1961), who established the importance of the parameter $\eta U/T$ in the motion of long bubbles in tubes, Taylor showed experimentally that the volume flow was dependent upon $\eta U/T$ when $h(c)$ was held constant. As we shall see this was indeed an important result.

### *Coyne–Elrod condition* (1970, 1971)

Coyne & Elrod argued that (2.3) holds throughout the lubrication regime and hence an appropriate condition to apply at the termination of this regime is

$$\partial p/\partial x = (6\eta U/h^2)(1-2\alpha) \tag{2.8}$$

provided that $\alpha = h_\infty/h(c)$ is known.

Essentially they looked at the steady fluid flow past the leading edge of the cavity as far as the uniform region at infinity (figure 3), from which they determined the ratios $h_\infty/h(c)$ and $r/h(c)$ as functions of $\eta U/T$:

$$h_\infty/h(c) = \alpha, \quad r/h(c) = \beta \tag{2.9}$$

(thereby confirming Taylor's results). Coyne & Elrod's solution to this difficult free-surface problem is not exact; it does involve assumptions regarding the wetting angle and the velocity distribution normal to the interface. Nevertheless for a cylinder–plane geometry all the experimentally known features are explained to a high degree of accuracy using (2.8) and (2.9).

Subsequently Smith (1975) sought to apply Coyne & Elrod conditions to a journal bearing of infinite width (figure 1*a*) whose gap thickness is given by

$$h(\theta) = h_0(1+e\cos\theta), \tag{2.10}$$

in which $x = R\theta$, $R$ is the radius of the rotating shaft and $h_0$ is the gap thickness when the shaft and bush are concentric. He found good agreement between theory and experiment for all $\eta U/T$ at high eccentricities (close to unity) but significant discrepancies for moderate eccentricities ($0{\cdot}3 < e < 0{\cdot}8$) and low values of $\eta U/T$. We shall offer an explanation for these discrepancies and examine further evidence all of which suggests that the Coyne & Elrod conditions are not strictly applicable to lightly loaded journal bearings.

Let $\bar{h}$ be the value of the gap thickness at which the pressure gradient is zero, then (2.3) gives $\bar{h} = 2h_\infty$ and (2.9) becomes

$$\bar{h}/h(c) = 2\alpha, \tag{2.11}$$

which is an expression which must hold whenever Coyne & Elrod's conditions are valid. Coyne & Elrod's results reveal that $\alpha$ decreases monotonically with decreasing $\eta U/T$, and for low values of $\eta U/T$, $\alpha$ is very much less than unity (e.g. for $\eta U/T = 0{\cdot}002$, $\alpha = 0{\cdot}015$).

In the particular range $\alpha \ll 1$ and for moderate eccentricities ($0{\cdot}3 < e < 0{\cdot}8$) the following simple argument clearly demonstrates that (2.11) can never be satisfied in a journal bearing. Since $h(\theta) = h_0(1+e\cos\theta)$ the ratio of the minimum gap thickness $h_{\min}$ to the maximum gap thickness $h_{\max}$ is

$$h_{\min}/h_{\max} = (1-e)/(1+e).$$

This ratio will always exceed $0{\cdot}1$ for the above eccentricities and in addition

$$\bar{h}/h(c) \geqslant h_{\min}/h_{\max}, \quad \text{thus} \quad \bar{h}/h(c) > 0{\cdot}1. \tag{2.12}$$

Clearly any $\bar{h}/h(c)$ satisfying (2.12) cannot simultaneously satisfy (2.11) when $\alpha \ll 1$, thereby precluding the application of Coyne & Elrod boundary conditions in this range. Further evidence to support this conclusion comes from observations in journal bearings over a wide range of operating conditions in which the cavity–fluid interface is seen to consist of a series of 'sharp-pointed fingers' separated by fluid. Such an interface does not fulfil the requirements of Coyne & Elrod's theory, which assumes the presence of a continuous cavity–fluid interface beneath which all the fluid flows to form a uniform layer of thickness $h_\infty$.

The elusive problem of ascertaining the correct boundary conditions at cavity formation in a journal bearing remains unsolved yet is worthy of further investigation because of its obvious practical importance.

## 3. Stability criterion for a cavity–fluid interface

As figure 2 demonstrates, a bearing of infinite width may exhibit a cavity–fluid interface whose shape in the $x$, $y$ plane may be either wavy or straight. In this section a simple stability argument is employed in order to establish a criterion for predicting the transition from linearity to waviness.

Without specifying any particular geometry we shall consider viscous fluid approaching the leading edge of a uniform cavity, and subsequently flowing beneath the cavity as shown in figure 2(*a*). In this equilibrium flow the force balance at the interface necessitates that the fluid pressure $P(c)$ together with the surface-tension pressure should balance the air pressure within the cavity, which is assumed to be atmospheric (zero):

$$P(c) + T/r = 0. \tag{3.1}$$

Naturally the interface $x = c$ is constantly subjected to small disturbances arising from fluctuations in fluid pressure within the bearing. An appropriate criterion, therefore, can be derived if we pose and answer the following question: under what conditions is the interface stable in the sense of having the capacity to resist disturbances of the kind mentioned above and remain uniform?

Figure 6 indicates the effect of one such disturbance in which the point $(c, y)$ on the straight interface is displaced to a new position $(c+\epsilon, y)$, where it is assumed that

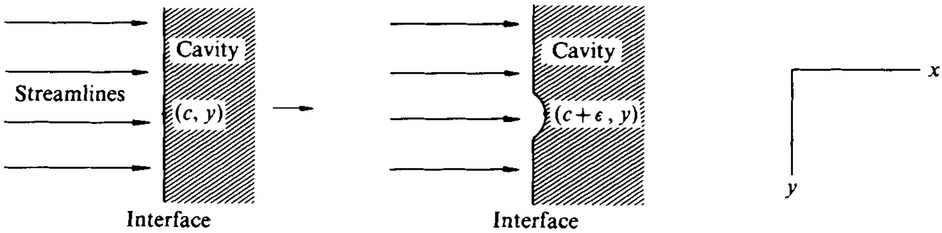


FIGURE 6. An $x, y$ cross-section of flow approaching a straight cavity–fluid interface in which the point $(c, y)$ is displaced to $(c+\epsilon, y)$.

$\epsilon/c \ll 1$. In the region of the perturbation the net force per unit area in the $x$ direction acting upon the interface (ignoring surface-tension effects in the $x, y$ plane) is

$$F = P(c+\epsilon)+T/r(c+\epsilon), \tag{3.2}$$

where $r(c+\epsilon)$ is the radius of curvature in the $x$, $z$ plane evaluated at $c+\epsilon$. Linearizing about $x = c$ and using (3.1) yields

$$F = \epsilon\frac{d}{dx}\left(P+\frac{T}{r}\right), \tag{3.3}$$

where the derivative is evaluated at $x = c$.

If the interface is stable to such disturbances it will return to its equilibrium position, in which case $F$ and $\epsilon$ must have opposite signs. The required criterion for the existence of a straight cavity–fluid interface is therefore

$$\frac{d}{dx}\left(P+\frac{T}{r}\right) < 0. \tag{3.4}$$

Equations (2.8) and (2.9) permit further simplification, and on writing $h_x$ for $\partial h/\partial x$, (3.4) becomes

$$h_x(c) > \frac{6\eta U}{T}\beta(1-2\alpha). \tag{3.5}$$

*Application to a cylinder–plane geometry*

In this geometry $h(x) = h_0(1+x^2/2Rh_0)$, in which $h_0 = h(0)$ is the minimum gap thickness. Hence, after making the transformation

$$x = (2Rh_0)^{\frac{1}{2}}\tan\gamma \quad [c = (2Rh_0)^{\frac{1}{2}}\tan\gamma^c],$$

one solves (2.3) subject to the Coyne & Elrod conditions (2.4) and (2.8) to obtain the relation

$$-\frac{T}{12\eta U\beta}\left(\frac{2h_0}{R}\right)^{\frac{1}{2}}\cos^2\gamma^c = \left(\frac{\gamma^c}{2}+\frac{\pi}{4}+\frac{\sin 2\gamma^c}{4}\right)+\left[\left(\frac{3\gamma^c}{8}+\frac{3\pi}{16}+\frac{\sin 2\gamma^c}{4}+\frac{\sin 4\gamma^c}{32}\right)\left(\frac{-2\alpha}{\cos^2\gamma^c}\right)\right]. \tag{3.6}$$

This equation essentillay determines the position of the cavity $\gamma^c$ once $\eta U/T$ and $h_0/R$ are fixed. When the transition from a straight to a wavy interface is required (3.5) provides a further relation among these three parameters, namely

$$\left(\frac{2h_0}{R}\right)^{\frac{1}{2}}\tan\gamma^c = 6\frac{\eta U}{T}\beta(1-2\alpha). \tag{3.7}$$

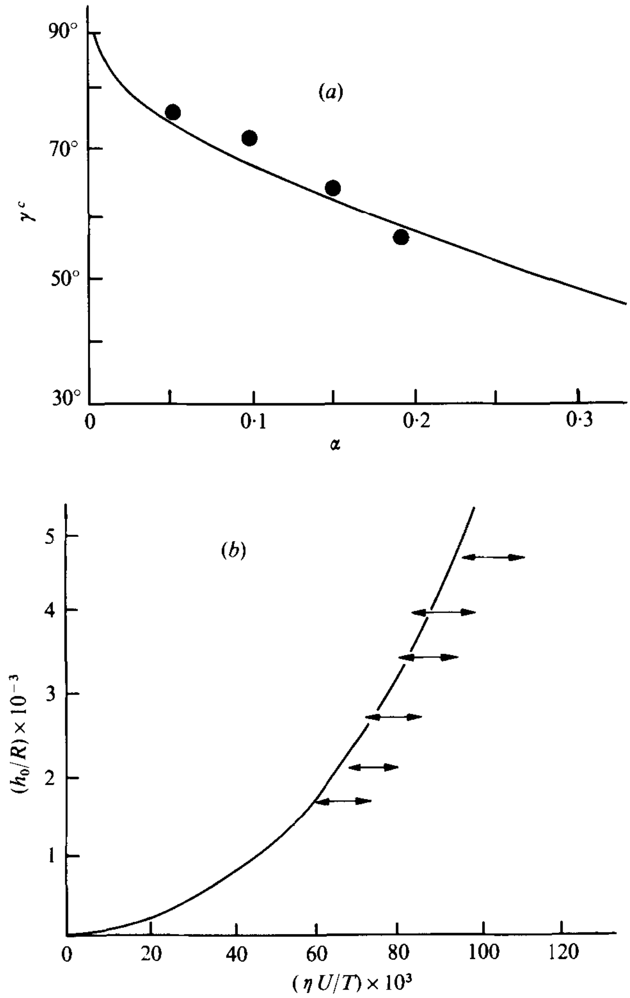


FIGURE 7. Theoretical curves of (*a*) $\gamma^c$ against $\alpha$ and (*b*) $h_0/R$ against $\eta U/T$. Experimental data: ●, Smith (1975); ↔, Savage.

Consequently $\gamma^c$ is obtained as a function of $\alpha$ and similarly $h_0/R$ is obtained as a function of $\eta U/T$ as shown in figures 7 (*a*) and (*b*), which also include comparison with experiment.

The cylinder–plane apparatus, which consisted of a Perspex block supported above a brass cylinder of radius 5 in. (figure 1*c*), permitted the range of values

$$1 < (h_0/R) \times 10^3 < 7$$

to be achieved. As figures 7(*a*) and (*b*) demonstrate, agreement between theory and experiment over this range is remarkably good.

### *Application to counter-rotating rollers*

A detailed treatment of the flow of thin liquid films between counter-rotating rollers, as shown in figure 1 (*b*), was given by Pitts & Greiller (1961). One of their objectives was to establish critical conditions to mark the transition from uniform flow past a uniform cavity to perturbed flow past a wavy interface using a theoretical model which assumes that the cavity shape in the $x, z$ plane close to the axis of symmetry is parabolic and of the form $z^2 = 4a(x-c)$. Taking the pressure drop across the cavity to be $T/2a$ they produce an argument similar to that described above to derive the stability criterion $d(P+T/2a)/dx < 0$, which, with a radius of curvature $r = 2a$, is identical to (3.4). Surprisingly however, they fail to stress the generality of either the method or the result, rather they concentrate on the determination of $a$ and $da/dx$ so as to conclude that at transition $(T/\eta U)\,h_0/R$ is approximately constant. Whilst their agreement between theory and experiment was good they were unable to satisfy the complete set of boundary conditions and in addition had to make several approximations, one of which assumes that $h(c)/h_0 = \sec^2\gamma^c$ varies only slowly with $\eta U/T$.

In contrast to the particular approach to this problem by Pitts & Greiller an alternative is that outlined in this paper using (3.4) together with the results of Coyne & Elrod. For this geometry it is convenient to define $h(x)$ to be half the gap thickness at a position $x$ along the axis of symmetry and then $h(x) = h_0(1+x^2/2Rh_0)$ as indeed is the case for a cylinder–plane geometry. Consequently the velocity distribution takes the form

$$u(x,z) = \frac{1}{2\eta}\frac{dP}{dx}(z^2-h^2)+U,$$

which in turn enables the continuity conditions to yield the following expression for the pressure gradient:

$$\frac{dP}{dx} = \frac{3\eta U}{h^2(x)}\left(1-\frac{h_\infty}{h(x)}\right). \tag{3.7a}$$

Writing $h_\infty/h(c) = \alpha$, the solution of (3.7$a$) which satisfies

$$P(-\infty) = 0, \quad P(c) = -\frac{T}{\beta h(c)}, \quad \left(\frac{dP}{dx}\right)_{x=c} = \frac{3\eta U}{h^2(c)}(1-\alpha)$$

is

$$-\frac{T}{6\eta U\beta}\left(\frac{2h_0}{R}\right)^{\frac{1}{2}}\cos^2\gamma^c = \left(\frac{\gamma^c}{2}+\frac{\pi}{4}+\frac{\sin 2\gamma^c}{4}\right)-\left(\frac{\alpha}{\cos^2\gamma^c}\right)\left(\frac{3\gamma^c}{4}+\frac{3\pi}{16}+\frac{\sin 2\gamma^c}{4}+\frac{\sin 4\gamma^c}{32}\right), \tag{3.7b}$$

in which $x = (2Rh_0)^{\frac{1}{2}}\tan\gamma^c$ represents the leading edge of the cavity. For transition to a wavy interface the stability criterion (3.4) provides the relation

$$\left(\frac{2h_0}{R}\right)^{\frac{1}{2}}\tan\gamma^c = \frac{3\eta U}{T}\beta(1-\alpha), \tag{3.7c}$$

thus permitting one to calculate $\gamma^c$ as a function of $\alpha$ and $h_0/R$ as a function of $\eta U/T$ as in the previous example of the cylinder–plane geometry. The curves so obtained are similar in shape over the whole domain to figures 7 ($a$) and ($b$) respectively and indeed this might have been expected from a comparison of (3.7$a$) and (3.7$b$) with (3.6) and (3.7), in which the only difference is a halving in magnitude of $\eta U$ and $\alpha$.

It is in the light of these theoretical curves that the results of Pitts & Greiller may be examined. Recall that their experiments were performed over a restricted range of $\eta U/T$ and $h_0/R$, well away from the limit of small $\eta U/T$ (small $\alpha$); then over such a restricted range $\gamma^c$ would vary slowly with $\eta U/T$ and $h_0/R$ would exhibit an almost linear variation with $\eta U/T$.

Finally, the limit of small $\eta U/T$ (small $h_0/R$) requires further experimental investigations, perhaps using rollers of larger diameter, so that the predicted behaviour may be confirmed or contradicted.

*Application to a journal bearing*

With the gap thickness given by $h(\theta) = h_0(1+e\cos\theta)$, (3.5) reduces to

$$\left(\frac{h_0}{R}\right)|e\sin\theta^c| > 6\frac{\eta U}{T}\beta(1-2\alpha), \tag{3.8}$$

where the position of cavity formation $\theta = \theta_c$ will lie in the range $\pi < \theta_c < 2\pi$ and for a given value of $\eta U/T$ the right-hand side of (3.8) will be a constant. Although, theoretically, it would appear that one can always find a value of $h_0/R$ which makes (3.8) valid and hence ensures the existence of a stable straight interface, it must be remembered that there is a limit to the magnitude of $h_0/R$ for which the assumptions of lubrication theory continue to hold. (In particular the flow in the $\theta, z$ plane must remain essentially one-dimensional and inertia effects be negligible.) The values of $h_0/R$ required by (3.8) are always in excess of this limit, thus precluding the presence of a straight cavity–fluid interface at cavity formation. Indeed something of the severity of this criterion for a journal can be appreciated by making a comparison with that for the cylinder–plane geometry, equation (3.7). The factor $h_0/R$ as opposed to $(h_0/R)^{\frac{1}{2}}$ means that under 'normal operating conditions' the left-hand side of (3.8) is at least an order of magnitude smaller.

Turning to the case of reformation, the stability argument has to be repeated and a new criterion determined, namely

$$\frac{d}{dx}\left(P+\frac{T}{r}\right) > 0 \quad \text{or} \quad \left(\frac{h_0}{R}\right)|e\sin\theta^r| < 6\frac{\eta U}{T}\beta(1-2\alpha). \tag{3.9}$$

Apart from $\theta^r$ replacing $\theta^c$ as the angular position of the interface, the sign of the inequality is reversed. It would indeed be convenient if one could deduce that (3.9) is necessarily valid since (3.8) is invalid. Strictly speaking this follows only if

$$|\sin\theta^r| \leqslant |\sin\theta^c|.$$

However, if we bear in mind that, under normal conditions, when (3.8) fails at formation (i.e. $(h_0/R)\,|e\sin\theta^c| < 6(\eta U/T)\beta(1-2\alpha)$) this is primarily due to the small magnitude of $h_0/R$, we see that in fact the slightly stronger inequality

$$\left(\frac{h_0}{R}\right)e < 6\frac{\eta U}{T}\beta(1-2\alpha) \tag{3.10}$$

would hold, in which case (3.9) would naturally be valid at reformation, thereby confirming the existence of a stable straight cavity–fluid interface.

The treatment given in this section is sufficient for predicting the transition from a straight to a wavy interface, but in order to understand the character of the perturbed interface the three-dimensional fluid flow in the vicinity of the interface has to be modelled and analysed. This forms the subject of part 2 of this study.

## REFERENCES

Bretherton, F. P. 1961 *J. Fluid Mech.* **10**, 166.
Coyne, J. C. & Elrod, H. G. 1970 *J. Lub. Tech.* **92**, 451.
Coyne, J. C. & Elrod, H. G. 1971 *A.S.M.E. Paper no.* 70 – Lub. 3.
Floberg, L. 1957 *Trans. Chalmers Univ. Tech.* no. 189.
Floberg, L. & Jakobson, B. 1957 *Trans. Chalmers Univ. Tech.* no. 190.
Hopkins, M. R. 1957 *Brit. J. Appl. Phys.* **8**, 442.
Pitts, E. & Greiller, J. 1961 *J. Fluid Mech.* **11**, 33.
Smith, E. 1975 Ph.D. thesis, University of Leeds.
Stieber, W. 1933 *Das Schwimmlager, V.D.I. (Berlin).*
Swift, H. W. 1932 *Proc. Inst. Civ. Engrs* **233**, 267.
Taylor, G. I. 1963 *J. Fluid Mech.* **16**, 595.
Van Der Bergh, H. 1974 M.Sc. project, Mech. Engng, University of Leeds.